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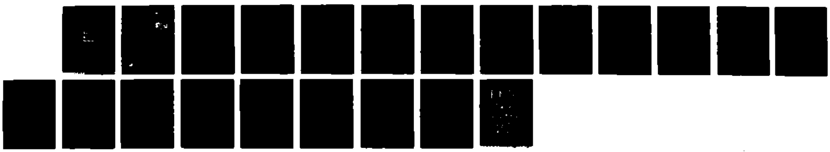

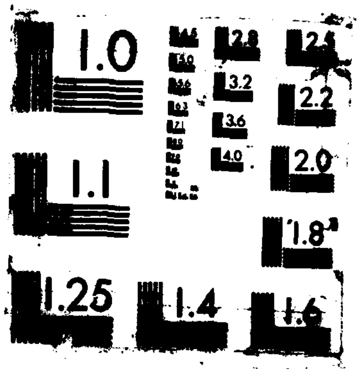
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**RECENT PROGRESS IN THE p AND h-p VERSION OF THE FINITE ELEMENT METHOD**

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The paper gives the state of the art and literature related to the development of the p and h-p version of the finite element method.

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# RECENT PROGRESS IN THE p AND h-p VERSIONS OF THE FINITE ELEMENT METHOD

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## 1. INTRODUCTION

The finite element method has become the main tool in computational mechanics. The success is manifested by the development of over five hundred user-oriented finite element program systems, ~~see e.g. [33]~~. The literature on the subject is overwhelming. To date there are over two hundred monographs and conference proceedings ~~[40]~~ and new monographs and proceedings are continuously appearing. Various forms of the finite element method are used in practice for the numerical treatment of elliptic, parabolic, hyperbolic, linear and nonlinear partial differential equations, integral and integrodifferential equations, etc. Any class of problems has its own specific features. ~~In~~ this paper ~~we will only~~ deals with the class of partial differential equation of elliptic type. For the sake of simplicity the author ~~we will~~ elaborates on a characteristic model problem and illustrative results and makes only additional comments of more general nature although the results ~~we refer~~ referred to are general.

## 2. THE MODEL PROBLEM AND ITS PROPERTIES

We restrict ourself to the most simple model problem. Let $\Omega \subset \mathbb{R}^2$, $x = (x_1,x_2)$ be a bounded (simple connected) domain with the boundary $\partial\Omega = \Gamma$ consisting of simple arcs

$$\Gamma_i = \{x_1 = f_{i,1}(t),\ \ x_2 = f_{i,2}(t),\ \ t \in I\}$$

where $I = (-1,1)$ and $\Gamma = \bigcup_{i=1}^{M} \bar{\Gamma}_i$. We will assume that $f_{i,1}$ and $f_{i,2}$ are analytic functions on $\bar{I}$ and $f'^2_{i,1} + f'^2_{i,2} > \alpha > 0$. The vertices of $\Omega$ will be denoted by $A_i$, $i = 1,\ldots,M$ and the internal angles at $A_i$ by $\omega_i$.

Let us be interested in the model problem and its standard (weak solution):

$$-\Delta u = f \text{ on } \Omega \tag{2.1}$$

$$u = \varphi \quad \text{on} \quad \Gamma^0 = \bigcup_{j \in Q} \bar{\Gamma}_j \tag{2.2a}$$

$$\frac{\partial u}{\partial n} = \psi \quad \text{on} \quad \Gamma^1 = \Gamma - \Gamma^0 \tag{2.2b}$$

Here $Q$ is a subset of $\{1,\ldots,M\} = \mathcal{M}$. For simplicity we assume that $\Gamma^0 \neq \emptyset$.

The performance of any numerical method strongly depends on the properties of the (exact) solution of the solved problem especially on its smoothness. The more information is available, the better method could be designed.

It is very advantageous to characterize the set of solutions of (2.1), (2.2) under consideration in the terms of countably normed spaces.

Let $A_i = (x_{1,i}, x_{2,i})$ and $r_i^2 = (x_1 - x_{1,i})^2 + (x_2 - x_{2,i})^2$. Define $\Phi_\beta(x) = \prod_{i=1}^{M} r_i^{\beta_i}(x)$, $\beta = (\beta_1, \ldots, \beta_M)$, $0 < \beta_i < 1$, and for any integer $k$ let $\Phi_{\beta \pm k}(x) = \prod_{i=1}^{M} r_i^{\beta_i \pm k}(x)$. Then we let $\mathcal{B}_\beta^2(\Omega) = \{u \in H^1(\Omega) \mid$

$\|\Phi_{\beta+k-2} D^\alpha u\|_{L_2(\Omega)} \leq C d^k k!$, $k = 2,3,\ldots$, $|\alpha| = k$, $C,d$ independent of $k\}$. If $u \in \mathcal{B}_\beta^2(\Omega)$ then it is analytic on $\bar{\Omega} - \bigcup_{i=1}^{M} A_i$ and has specific behavior in the neighborhood of $A_i$, $i = 1,\ldots,M$. In [7] [8] we have proven

Theorem 2.1. Let $f$ be analytic on $\bar{\Omega}$, $\varphi$ be analytic on $\bar{\Gamma}_j$, $j \in Q$ and continuous $\bar{\Gamma}^0$, $\psi$ be analytic on $\Gamma_j$, $j \in \mathcal{M} - Q$. Then for (2.1) and (2.2) $u \in \mathcal{B}_\beta^2(\Omega)$ with $\beta_i > \bar{\beta}_i$; $\beta_i$ depends on $\omega_i$ and the type of boundary condition on $\Gamma_{i-1}$ and $\Gamma_i$. □

If, for example, $M = 5$, $Q = \{1,2,3\}$ we get $\bar{\beta}_1 = \max(0, 1 - \frac{\pi}{4\omega_1})$, $\bar{\beta}_2 = \max(0, 1 - \frac{\pi}{2\omega_2})$, $\bar{\beta}_3 = \max(0, 1 - \frac{\pi}{2\omega_3})$, $\bar{\beta}_4 = \max(0, 1 - \frac{\pi}{4\omega_4})$, $\bar{\beta}_5 = \max(0, 1 - \frac{\pi}{2\omega_5})$.

Remark 2.1. In [9] we precisely characterized the traces of functions from $\mathcal{B}_\beta^2(\Omega)$ and gave full characterization of the sets of $f$, $\varphi$ and $\psi$ which guarantee that the solution $u$ of (2.1) (2.2) belongs to $\mathcal{B}_\beta^2$.

Remark 2.2. Theorem 2.1 also holds when the differential equation in (2.1) has analytic coefficients on $\bar{\Omega}$ (see [7]).

Remark 2.3. The eigenfunctions of the eigenvalue problem (2.1) (2.2) also belong to $B^2_\beta(\Omega)$ (see [10]).

Remark 2.4. Theorem 2.1 also holds for strongly elliptic system of differential equations as elasticity equations (see [11]).

In practice, e.g. in the field of structural mechanics, the problems of partial differential equations are characterized by piecewise analytic data and hence theorem 2.1 is very well suited for the applications.

## 3. THE FINITE ELEMENT METHOD

There are various different forms of the finite element method. We will consider here only the basic class of finite element methods (for our model problem).

Let $T = \{\tau_i\}$ be a partition of $\Omega$ into (in general curvilinear) triangles or quadrilaterals called elements $\tau_i$. In the case when $T$ is a triangulation we are making the standard assumptions. For the general case we refer to [8], [9], [36]. We will formulate here the assertions in the case of triangulation only although they hold in general. Let $H(p,T) = \{u \in H^1(\Omega) | u|_{\tau_i}, \tau_i \in T$ is a polynomial of degree $p\}$ be the finite element space. If $\tau_i$ is a rectangle, then polynomials are of degree $p$ in both variables. If the elements are curvilinear, then $u|_{\tau_i}$ are the standard "pull-back" polynomials.

Remark 3.1. We have assumed that the degree of polynomials are the same over all elements. The theory is developed for general case when the degree $p$ can be different on different elements.

Let $H_0(p,T) = H(p,T) \cap H^1_0(\Omega)$ where $H^1_0(\Omega) = \{u \in H^1(\Omega) | u = 0$ on $\Gamma^0\}$ and $H_0(p,T)$ be the restriction of $H(p,T)$ on $\Gamma^0$. We will assume that a projection operator $P_0(p,T)$ of function $\varphi$ into $H_0(p,T)$ be given and we denote $\varphi_{p,T} = P_0(p,T)\varphi$. A concrete possible form of $P_0(p,T)$ will be given later.

The finite element method consists now in finding $u_{FE} = u(p,T) \in H(p,T)$ so that

1) $u(p,T) = \varphi_{p,T}$ on $\Gamma^0$

2) $$\iint_\Omega \left(\frac{\partial u_{FE}}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial u_{FE}}{\partial x_2}\frac{\partial v}{\partial x_2}\right)dx_1 dx_1$$

$$= \int_{\Gamma^1} \psi v \, ds + \iint_\Omega f v \, dx_1 dx_2$$

holds for any $v \in H_0(p,T)$.

We will be interested in the accuracy of the finite element solution measured in the energy norm. Define $e = u - u_{FE}$ and let

$$\|e(T,p)\|_E^2 = \iint_\Omega \left(\left(\frac{\partial e}{\partial x_1}\right)^2 + \left(\frac{\partial e}{\partial x_2}\right)^2\right)dx_1 dx_2$$

be the error measured in the energy norm.

Two kind of operators $P_0(T,p)$ can be considered. Let $\gamma \subset \Gamma^0$ be the side of the triangle $\tau$ with endpoints $A,B$ and assume that $\gamma = [-1,1]$. $A = -1$, $B = 1$. Then $\varphi_{p,T}|_\gamma = \ell(x) + w(x)$ where $\ell(x)$ is linear function on $\gamma$ such that $\varphi_{p,T}|_\gamma(\pm 1) = \varphi(\pm 1)$ and a) in the case of $H^1$-projection: $P_0^1(\tau,p)$: $\varphi' = \sum_{k=0}^{\infty} a_k \ell'_k$, $w'(x) = \sum_0^{p-1} a_k \ell'_k$, where $\ell_k$ are the Legendre polynomials; b) in the case of $H^{1/2}$-projection $P_0^{1/2}(T,p)$: $\varphi(x) = \sum_{k=0}^{\infty} a_n T_k(x)$, $w(x) = \sum_{k=0}^{p} a_k T_k(x)$, where $T_k(x)$ are the Tchebyschev polynomials.

As we have seen there is a large freedom in the selection of $H(p,T)$ namely the degree $p$ and the partition $T$ and in the selection of the operators $P_0(T,p)$. We expect that $\|e\|_E \to 0$ if $\dim H_0(p, \ ) \to \infty$.

It is convenient to distinguish in this context three versions of the finite element method.

a) The h-version. Here a sequence (family) $H(p,T_i)$ is considered when $p$ is fixed (usually $p = 1,2$) and the mesh $T_i$ is successively refined so that the size $h$ of the elements of $T_i$ goes to zero.

b) The p-version. Here the mesh $T$ is kept fixed and $p \to \infty$ uniformly on selectively.

c) The h-p version. In this version the mesh is simultaneously refined and the degree p increased uniformly or selectively.

The h-version of the finite element method is the standard one and extensive literature is devoted to it. The p-version is a recent development. The first theoretical paper about the p-version [26] and the h-p version [6] appeared in 1981 and various results were obtained since then. Some of them will be mentioned later.

There are many codes, research and commercial utilizing the h-version. The only commercial code using p and h-p versions is the code PROBE which was developed recently by NOETIC Technologies, St. Louis [54]. PROBE solves two dimensional problems of linear elasticity, stationary heat problems and thermoelasticity problems. The three dimensional extension of PROBE will be released in 1988. Three dimensional research code STRIPE was developed by Swedish Aeronautical Research Institute, see e.g. [1]. These codes have in addition various features as adaptive approaches, various a-posteriori error estimation, etc.

## 4. THE BASIC PROPERTIES OF THE p AND h-p VERSIONS OF THE FINITE ELEMENT METHOD

After 1980 the p and h-p versions of the finite element method was studied in detail from various point of view. We will mention here some essential illustrative results.

In one dimensional setting the versions were analyzed in detail in [34]. Here, among other, the optimal meshes and p-distribution has been established with upper and lower bounds of the errors for the three basic finite element versions.

In two dimensional setting the following theorem is characteristic for the performance of the h-p version. (For details see [8], [9], [36].)

Theorem 4.1. Let the solution u of the problems (2.1), (2.2) belongs to the set $B^2_\beta(\Omega)$. Then there is a sequence of meshes $T_i$ and the degrees $p_i$ such that

$$\|e\|_E < C e^{-\alpha\sqrt[3]{N_i}}, \qquad \alpha > 0 \tag{4.1}$$

where $N_i = \dim H_0(p_i, T_i)$ is the number of degrees-of-freedom for the h-p version. □

In one dimension the rate is $Ce^{-\alpha\sqrt[2]{N}}$. It has been proven in [34] that the optimal mesh is a geometric one with the factor $(\sqrt{2}-1)^2 = .17$. The experience shows that the geometric mesh with the factor $= .15$ is also optimal in two dimension. Theorem 4.1 together with Theorem 2.1 shows that practically in any problem of structural mechanics the exponential rate of convergence can be achieved.

For the p-version the following theorem is another typical one (for more, see [21], [22], [23], [24]).

Theorem 4.2. Let us consider the problem (2.1) (2.2) and let $\bar{\beta} = \max(\bar{\beta}_i)$ given in Theorem 2.1. Then for the p-version we have

$$\|e\|_E \leq CN^{-(1-\bar{\beta})} \tag{4.2}$$

while for the h-version with uniformed mesh

$$\|e\|_E \geq CN^{-\left(\frac{1-\bar{\beta}}{2}\right)} . \tag{4.3}$$

□

In Theorems 4.1 and 4.2 either the projection $P_0^1$ or $P_0^{1/2}$ could be used provided $\varphi$, $\psi$ are sufficiently smooth. The difference between these two operators occurs for the p-version when the boundary condition $\varphi$ is unsmooth, e.g. $\varphi \in H^\delta(\Gamma)$, $\frac{1}{2} < \delta < 3/4$. In this case the $H^{1/2}$ projection has to be used. For the h-p version there is no difference in the asymptotic rate but some difference occurs in the constant of the estimates. For the analysis of the influence of the operator $P_0$ on the accuracy we refer to [12].

Remark 4.1. So far we have assumed that the domain $\Omega$ is bounded. Nevertheless the exponential rate of convergence (4.1) holds also for the problem on $\Omega^C = R^2 - \Omega$ when $f$ has bounded support. Here the infinite elements and properly selected shape functions have to be used. For more, see [15].

Remark 4.2. The h, p and h-p versions have different aspects with respect to the pollution problem. In presence of a singular behaviour of the solution (e.g. in the neighborhood of the entrant corner of the domains) the $L_\infty$ error is very large in the element consisting the corner. This effect disappears in elements which are separated away from the singularity by few elements. This effect is essential for a proper mesh design in practical computation. For details we refer to [14].

So far we have dealt with the problem (2.1) and (2.2) of second order. For the analyses of the finite element solution for the problems of order $2k$ we refer in the case of the h-p version to [35] and the p-versi n to [39] [53]. For the basic analysis of the p-version in 3 dimensions, we refer to [30] and [31]. The eigenvalue problem is, as is well known, directly related to the "source" problem we addressed earlier. See e.g. [16] and [17]. In the case of the eigenvalue problem (in our case)

$$\Delta u = \lambda u$$

$$u = 0 \quad \text{on} \quad \Gamma^0$$

$$\frac{\partial u}{\partial u} = 0 \quad \text{on} \quad \Gamma^{(1)}.$$

The eigenfunctions belong to $B^2_\beta(\Omega)$ and hence

$$|\lambda - \lambda_{FE}| \leq C\, e^{-2\alpha\sqrt[3]{N}}$$

$$\|u_{FE}(\lambda_h) - u(\lambda)\|_E \leq C\, e^{-\alpha\sqrt[3]{N}}.$$

For more details, see e.g. [10].

## 5. IMPLEMENTATION AND COMPUTATIONAL COMPLEXITY

There are some essential features of implementation of the p and h-p versions. See e.g. [56]. The elements are of a hierarchical type which leads to augmentation (bordering) of the local stiffness matrices when p is increased. This also allows to change very flexibly the degree of the shape functions from one element to the other one. The shape elements are (in two dimensions) of nodal type, side type and internal type and are based (in PROBE) on the integrals fo the Legendre polynomials. This is important for numerical stability aspects. The computation (in two dimension) of the local stiffness matrix on a rectangle requires (when optimally programmed) $O(p^4)$ operations. The computation of the local stiffness matrices takes much more effort for the p-version (with high p) that for the h-version and hence the p-version is very well suited for parallel computations. The sparsity of the global stiffness matrix is also smaller for the p version than the h-version. Hence, the complexity of the computation for the same number of degrees-of-freedom is higher for the p-version than the h-version. Nevertheless, it is essential to relate the achieved accuracy to the computational effort. For an analysis of the computational complexity and computer time comparison, we refer to [19] and [20]. The results show that the h-p version with higher degree p is preferable for solutions which are not overly unsmooth or have singular behaviour in a-priori known areas as in the neighborhood of the corners. If the solution is very unsmooth, e.g. if the coefficients in the differential equations are uniformly rough, for example, measurable only, then only low accuracy is practically achievable by any method and h-version with low p is preferable. In general, for very low accuracies the low order elements are preferable, for the modest one higher degree elements have to be preferred.

## 6. THE PROBLEM OF THE MESH DESIGN

One of the most laborious part of the finite element analysis, especially in three dimensions, is the mesh generation also when sophisticated mesh generators are used. The use of large elements (possibly of high degrees) which are describing only the geometry, greatly simplifies the user's work, also if possibly on expense of the computer time. (It is necessary to realize that the relation between manpower cost and computation cost is going steadily up.) The option of a change of the degrees of elements increases significantly the flexibility of the program and gives the user effective tool for the quality control. The p and h-p version programs give such options. It is advantageous therefore to create directly or indirectly the proper mesh and to achieve then the desired accuracy by an appropriate choice of the degrees, which can be made, for example, in an adaptive mode. The goal is to achieve the same combination of the degrees and mesh refinement which would be obtained for the given accuracy by the h-p version directly. To achieve this goal two avenues could be followed, the expert system and the adaptive approach. The expert system, see e.g. [18], [45], advises the user how to design the mesh and element degrees for the requested accuracy and provides the user with a mesh generator. The expert system is interactive, follows the progress of computation, gives the user on his request various desired information for an effective computation and engineering analysis. The experience (see [18] [19]) is that the cost of the expert activity for a mesh and degree design is at most 20% of the total cost.

The adaptive approach (see e.g. [46], [34], Part 3) which is possible to see as an "automatic" expert system makes various decisions for the users. Both approaches have some common parts but the concepts are significantly different. We refer also to [46] for various additional aspects.

## 7. THE ROBUSTNESS

An effective method has to perform uniformly well for a broad class of input data. The elasticity problems can be in practice nearly singular as, for example, in the case of nearly incompressible material, various plate and shell theories, in the case of thin domains, etc. The h-version suffers in these cases by the "locking" problem which has to be overcome by various special approaches as reduced integration, etc. (see e.g. [28]). Problems of these types are usually avoided when the p and h-p versions are used. The convergence rate then (in contrast to the h-versions) is uniform with respect to the Poisson ratio when higher degrees of elements are used. See e.g. [49], [50] [64].

## 8. THE QUALITY CONTROL OF THE SOLUTION

It is essential to have a possibility of a quantitative assessment of the quality of computed data. For the survey of today's general ideas and results in this direction, we refer to [41]. In the case of the p and h-p versions there is relatively easy way for the quality assessment of any data of the interest by changing the degrees and by an extrapolation procedure. This approach is very effective because it indicates reliably the errors of any computed data of interest, the energy norm, value of stresses, stress intensity factor, etc. See e.g. [1], [8], [36], [58], [61].

## 9. THE COMPUTATIONAL AND ENGINEERING EXPERIENCE

Because of the developed commercial code PROBE and research code STRIPE an extensive experience is already available in the research and industrial use. For the industrial experience we refer e.g. to [27] where numerical results are presented. For the research one we refer e.g. to [1], [3], [8],

[9], [10], [18], [19], [20], [25], [36], [48], [55], [57], [63]. The experience shows that the p and h-p versions of the finite element method has many practical advantages for the engineering computations for linear elliptic problems.

## 10. RELATION TO SOME OTHER METHODS

The ideas of the p and h-p versions are related for example to the various methods used in fluid dynamics as the spectral method and its variation. We refer e.g. to [43] and references given there. A commercial fluid dynamics code written by T. Patera is closely related to the idea of h-p version of the finite element method.

The h-p version of course generates various finite element approaches and in principle encompasses various diverse approaches, see e.g. [29]. The h-p version can be naturally also used for solving integral equations, boundary element method, etc. See e.g. [47], [51], [67].

## 11. SUMMARY

The p and h-p version of the finite element method is a new development which gives new possibilities, theoretical and practical for the finite element method. It is today reasonably well understood in the case of elliptic equations, both theoretically and practically. The aim of the present paper was to give a brief survey of various aspects of the p and h-p versions of the finite element method for solving elliptic linear problems and provide the comprehensive references. Nevertheless, many theoretical and practical aspects of the method for other problems, linear and nonlinear are still to be resolved as well as the problems of implementation for three dimensional problems.

In addition we refer to the references [2], [3], [4], [37], [38], [42], [44], [52], [58], [60], [62], [65], [66] which are directly related to the subject discussed.

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